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Linear Algebra and its Applications 342 (2002) 41–46

**LINEAR ALGEBRA
AND ITS
APPLICATIONS**

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A note on the variation of the spectrum of an arbitrary matrix

Yongzhong Song

*School of Mathematics and Computer Science, Nanjing Normal University,
Nanjing 210097, People's Republic of China*

Received 7 May 2001; accepted 7 July 2001

Submitted by L. Elsner

Abstract

Let A and B be two $n \times n$ matrices with spectra $\lambda(A) = \{\lambda_1, \dots, \lambda_n\}$ and $\lambda(B) = \{\mu_1, \dots, \mu_n\}$. Suppose that the nonsingular matrix Q satisfies $Q^{-1}AQ = \text{diag}(J_1, \dots, J_p)$, where each submatrix J_i , $i = 1, \dots, p$, is a Jordan block. Then there exists a permutation π of $\{1, \dots, n\}$ such that

$$\sqrt{\sum_{j=1}^n |\mu_{\pi(j)} - \lambda_j|^2} \leq \sqrt{n}(1 + \sqrt{n-p}) \max \left\{ \|Q^{-1}(B-A)Q\|_F, \sqrt[m]{\|Q^{-1}(B-A)Q\|_F} \right\}$$

and for $j = 1, \dots, n$,

$$|\mu_{\pi(j)} - \lambda_j| \leq \sqrt{n}(1 + \sqrt{n-p}) \max \left\{ \sqrt{n} \|Q^{-1}(B-A)Q\|_2, \sqrt[m]{\sqrt{n} \|Q^{-1}(B-A)Q\|_2} \right\},$$

where m is the order of the largest Jordan block of A and $\|\cdot\|_F$ and $\|\cdot\|_2$ denote, respectively, the Frobenius norm and the spectral norm. © 2002 Elsevier Science Inc. All rights reserved.

AMS classification: 65F15

Keywords: Matrix; Spectrum; Variation

E-mail address: yzsong@pine.njnu.edu.cn (Y. Song).

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1. Introduction

Let A and B be two $n \times n$ matrices with spectra $\lambda(A) = \{\lambda_1, \dots, \lambda_n\}$ and $\lambda(B) = \{\mu_1, \dots, \mu_n\}$. The bound of the “distance” between two spectra $\lambda(A)$ and $\lambda(B)$ has been investigated in many papers and books.

When A and B are both normal matrices, Hoffman and Wielandt [1] proved that there exists a permutation π of $\{1, \dots, n\}$ such that

$$\sqrt{\sum_{j=1}^n |\mu_{\pi(j)} - \lambda_j|^2} \leq \|B - A\|_F,$$

where $\|\cdot\|_F$ denotes the Frobenius norm.

Furthermore, Sun [2] proved that if A is normal and B is nonnormal, then there exists a permutation π of $\{1, \dots, n\}$ such that

$$\sqrt{\sum_{j=1}^n |\mu_{\pi(j)} - \lambda_j|^2} \leq \sqrt{n} \|B - A\|_F \quad (1.1)$$

and

$$|\mu_{\pi(j)} - \lambda_j| \leq n \|B - A\|_2, \quad j = 1, \dots, n. \quad (1.2)$$

An example in [2] shows that the factor \sqrt{n} in (1.1) is best possible.

In this note, we investigate the case when A and B are arbitrary matrices. Suppose that the nonsingular matrix Q satisfies

$$Q^{-1}AQ = \text{diag}(J_1, \dots, J_p),$$

where J_i is a Jordan block, $i = 1, \dots, p$. With the aid of [2, Theorem 1.1] we will prove that, in this case, there exists a permutation π of $\{1, \dots, n\}$ such that

$$\sqrt{\sum_{j=1}^n |\mu_{\pi(j)} - \lambda_j|^2} \leq \sqrt{n}(1 + \sqrt{n-p}) \max \left\{ \|Q^{-1}(B - A)Q\|_F, \sqrt[m]{\|Q^{-1}(B - A)Q\|_F} \right\}$$

and for $j = 1, \dots, n$,

$$|\mu_{\pi(j)} - \lambda_j| \leq \sqrt{n}(1 + \sqrt{n-p}) \times \max \left\{ \sqrt{n} \|Q^{-1}(B - A)Q\|_2, \sqrt[m]{\sqrt{n} \|Q^{-1}(B - A)Q\|_2} \right\},$$

where m is the order of the largest Jordan block of A and $\|\cdot\|_F$ and $\|\cdot\|_2$ denote, respectively, the Frobenius norm and the spectral norm.

2. Main results

For any matrix A , there exists a nonsingular matrix Q transforming A into its Jordan canonical form

$$Q^{-1}AQ = J = \begin{pmatrix} J_1 & & \\ & J_2 & \\ & & \dots \\ & & & J_p \end{pmatrix}, \quad (2.1)$$

where, for $i = 1, \dots, p$, $J_i \in \mathcal{C}^{k_i \times k_i}$ is a Jordan block with the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \ddots & & \\ & & \lambda_i & 1 \\ & & & \lambda_i \end{pmatrix}.$$

For $\varepsilon \neq 0$ and $i = 1, \dots, p$, let $T_i \in \mathcal{C}^{k_i \times k_i}$ be a diagonal matrix with the form

$$T_i = \text{diag}(1, \varepsilon, \varepsilon^2, \dots, \varepsilon^{k_i-1}). \quad (2.2)$$

Then T_i is nonsingular and it holds

$$T_i^{-1}J_iT_i = \begin{pmatrix} \lambda_i & \varepsilon & & \\ & \ddots & & \\ & & \lambda_i & \varepsilon \\ & & & \lambda_i \end{pmatrix}.$$

Denote

$$T = \text{diag}(T_1, \dots, T_p). \quad (2.3)$$

Then T is nonsingular and we have

$$T^{-1}Q^{-1}AQ T = \begin{pmatrix} T_1^{-1}J_1T_1^{-1} & & & \\ & T_2^{-1}J_2T_2^{-1} & & \\ & & \dots & \\ & & & T_p^{-1}J_pT_p^{-1} \end{pmatrix} = A + \Omega$$

with

$$A = \begin{pmatrix} \lambda_1 I & & & \\ & \lambda_2 I & & \\ & & \dots & \\ & & & \lambda_p I \end{pmatrix}, \quad \Omega = \begin{pmatrix} \Omega_1 & & & \\ & \Omega_2 & & \\ & & \dots & \\ & & & \Omega_p \end{pmatrix}$$

and

$$\Omega_i = \begin{pmatrix} 0 & \varepsilon & & \\ & 0 & \varepsilon & \\ & & \ddots & \\ & & & 0 & \varepsilon \\ & & & & 0 \end{pmatrix}, \quad i = 1, \dots, p.$$

We now prove the main statement.

Theorem 2.1. *Let A and B be arbitrary matrices. Suppose that the nonsingular matrix Q satisfies (2.1). Then there exists a permutation π of $\{1, \dots, n\}$ such that*

$$\sqrt{\sum_{j=1}^n |\mu_{\pi(j)} - \lambda_j|^2} \leq \sqrt{n}(1 + \sqrt{n-p}) \max \left\{ \|Q^{-1}(B-A)Q\|_F, \sqrt[m]{\|Q^{-1}(B-A)Q\|_F} \right\}, \quad (2.4)$$

where m is the order of the largest Jordan block in (2.1).

Proof. Denote

$$E = B - A.$$

Assume that

$$\|Q^{-1}EQ\|_F \leq 1.$$

Then let

$$\varepsilon = \sqrt[m]{\|Q^{-1}EQ\|_F}$$

and let T be defined by (2.2) and (2.3). Then

$$\|T\|_2 = 1, \quad \|T^{-1}\|_2 = \varepsilon^{1-m}$$

and

$$T^{-1}Q^{-1}EQT + \Omega = T^{-1}Q^{-1}BQT - A.$$

Since the matrix A is normal and the matrix $T^{-1}Q^{-1}BQT$ has the same spectrum with B , by [2, Theorem 1.1] we obtain that there exists a permutation π of $\{1, \dots, n\}$ such that

$$\begin{aligned} \sqrt{\sum_{j=1}^n |\mu_{\pi(j)} - \lambda_j|^2} &\leq \sqrt{n} \|T^{-1}Q^{-1}EQT + \Omega\|_F \\ &\leq \sqrt{n} (\|T\|_2 \|T^{-1}\|_2 \|Q^{-1}EQ\|_F + \|\Omega\|_F) \\ &\leq \sqrt{n} (\varepsilon^{1-m} \|Q^{-1}EQ\|_F + \varepsilon \sqrt{n-p}) \\ &= \sqrt{n}(1 + \sqrt{n-p}) \sqrt[m]{\|Q^{-1}(B-A)Q\|_F}. \end{aligned} \quad (2.5)$$

We now assume that

$$\|Q^{-1}EQ\|_F \geq 1.$$

Let $\varepsilon = 1$ and $T = I$. Then we have

$$Q^{-1}EQ + \Omega = Q^{-1}BQ - A$$

and there exists a permutation π of $\{1, \dots, n\}$ such that

$$\begin{aligned} \sqrt{\sum_{j=1}^n |\mu_{\pi(j)} - \lambda_j|^2} &\leq \sqrt{n} \|Q^{-1}EQ + \Omega\|_F \\ &\leq \sqrt{n} (\|Q^{-1}EQ\|_F + \|\Omega\|_F) \\ &\leq \sqrt{n} \left(\|Q^{-1}EQ\|_F + \sqrt{n-p} \right) \\ &\leq \sqrt{n} (1 + \sqrt{n-p}) \|Q^{-1}(B-A)Q\|_F. \end{aligned} \quad (2.6)$$

By (2.5) and (2.6) we have proved (2.4). \square

Since the inequality $\|X\|_F \leq \sqrt{n} \|X\|_2$ holds for any matrix X , from Theorem 2.1 it derives the following result, immediately.

Corollary 2.2. *Let A and B be arbitrary matrices. Suppose that the nonsingular matrix Q satisfies (2.1). Then there exists a permutation π of $\{1, \dots, n\}$ such that*

$$\begin{aligned} |\mu_{\pi(j)} - \lambda_j| &\leq \sqrt{n} (1 + \sqrt{n-p}) \\ &\quad \times \max \left\{ \|Q^{-1}(B-A)Q\|_F, \sqrt[m]{\|Q^{-1}(B-A)Q\|_F} \right\} \\ &\leq \sqrt{n} (1 + \sqrt{n-p}) \\ &\quad \times \max \left\{ \sqrt{n} \|Q^{-1}(B-A)Q\|_2, \sqrt[m]{\sqrt{n} \|Q^{-1}(B-A)Q\|_2} \right\}, \\ &\quad j = 1, \dots, n. \end{aligned}$$

We denote the Jordan condition number of A by κ , namely,

$$\kappa = \inf_Q \{ \|Q\|_2 \|Q^{-1}\|_2, Q^{-1}AQ = J \}.$$

Clearly, for any matrices X and Y , it holds $\|XY\|_F \leq \|X\|_2 \|Y\|_F$. Then from Theorem 2.1 and Corollary 2.2 we directly get the following result.

Corollary 2.3. *Let A and B be arbitrary matrices. Suppose that the nonsingular matrix Q satisfies (2.1). Then there exists a permutation π of $\{1, \dots, n\}$ such that*

$$\sqrt{\sum_{j=1}^n |\mu_{\pi(j)} - \lambda_j|^2} \leq \sqrt{n} (1 + \sqrt{n-p}) \max \left\{ \kappa \|B-A\|_F, \sqrt[m]{\kappa \|B-A\|_F} \right\}$$

and for $j = 1, \dots, n$,

$$|\mu_{\pi(j)} - \lambda_j| \leq \sqrt{n}(1 + \sqrt{n-p}) \max \left\{ \kappa \sqrt{n} \|B - A\|_2, \sqrt[m]{\kappa \sqrt{n} \|B - A\|_2} \right\}.$$

We now consider the case when the matrix A is diagonalizable. In this case, there exists a nonsingular matrix Q such that

$$Q^{-1}AQ = A = \text{diag}(\lambda_1, \dots, \lambda_n).$$

This shows that, in this case, we have $p = n$ in (2.1) and $m = 1$. From Theorem 2.1, Corollaries 2.2 and 2.3, the following result follows.

Theorem 2.4. *Let A be a diagonalizable matrix. Suppose that the nonsingular matrix Q satisfies*

$$Q^{-1}AQ = A.$$

Then there exists a permutation π of $\{1, \dots, n\}$ such that

- (i) $\sqrt{\sum_{j=1}^n |\mu_{\pi(j)} - \lambda_j|^2} \leq \sqrt{n} \|Q^{-1}(B - A)Q\|_F$;
- (ii) $\sqrt{\sum_{j=1}^n |\mu_{\pi(j)} - \lambda_j|^2} \leq \kappa \sqrt{n} \|B - A\|_F$;
- (iii) $|\mu_{\pi(j)} - \lambda_j| \leq n \|Q^{-1}(B - A)Q\|_2, \quad j = 1, \dots, n$;
- (iv) $|\mu_{\pi(j)} - \lambda_j| \leq n\kappa \|B - A\|_2, \quad j = 1, \dots, n$.

When A is normal from here we can derive (1.1) and (1.2).

References

- [1] A.J. Hoffman, H.W. Wielandt, The variation of the spectrum of a normal matrix, *Duke Math. J.* 20 (1953) 37–39.
- [2] J.-G. Sun, On the variation of the spectrum of a normal matrix, *Linear Algebra Appl.* 246 (1996) 215–223.